RICHARD, BERRY AND CONCEPT EXTENSIBILITY

I point out some difficulties in the most accepted proposal of solution to Richard's and Berry's paradoxes and offer a new approach based on the concept of indefinite extensibility and the distinction between signifier and signified as introduced by Saussure. By applying set theoretical concepts to linguistic (semantical) objects a new interpretation of the concept of 'set-like collection' is suggested. By relating extensibility to hierarchies of logical levels I also sketch a similitude between a form of tokenism and extensibility.

Key words: Berry's paradox; diagonalization; enumerability; extensibility; fuzziness; logical level; Richard's paradox; set; signified; signifier; tokenism.

I. RICHARD

Shapiro et al. (2006: 266) have recently introduced the concept of indefinite extensibility of a concept P with respect to some concept Q. Let me put in a simple way: a concept P is indefinitely extensible with respect to the concept Q if and only if the following two conditions hold:

1. There is a function *f* such that for any set S of P's, if S is Q, then f(S) yields a P not in S.

2. $S \cup \{f(S)\}$ is Q.

The consequence is always that there is no set that is Q and contains all P's.

The simplest example is the concept P' of natural number, which is indefinitely extensible with relative to the concept Q' of finite, that is, for any finite set S' of natural numbers there is a function *f'* that gives a natural not in S' and such that $S' \cup \{f'(S')\}$ is finite; f'(S') is, for instance, the successor of the greatest natural in S'. The consequence is that there is no finite set containing all natural numbers. Similarly the concept P'' of real number is indefinitely extensible with respect to the concept Q'' of enumerable, being *f''* the Cantorian diagonalization procedure.

An extreme example is the concept of set being indefinitely extensible with respect to the concept of set, so that there is no set of all sets.

Let's consider an ideal speaker SP of English. By 'ideal' I mean that SP won't be hindered by physical or psychological limitations in time, space, health, memory, supply of paper... and that SP will be able to use all resources of English and all possible human mathematical knowledge, and that SP will be a consistent 'cumulative' speaker at any time and over time. SP starts with consistent rules and can certainly innovate and create new expressions by definition from the ones he already has, as well as the necessary rules to use them, but he will neither reject any previously accepted expression or linguistic rule nor introduce any rule contradicting another previous rule. It suffices if the existence of such a speaker is a logical possibility, as it seems to be.

SP can define in English some sets of naturals like the set N of all naturals, the set of all naturals except 0, the set of all naturals except 0 and the successor of 0, and so on, which gives us an infinite supply of definite English definitions of sets of naturals. Of course, there are definitions of fuzzy sets of naturals, like 'the set of numbers that count the mountains in each country of the world'; there may also be some sequences of words that are only definitions up to some degree; for instance: ‘the set of any natural greater than 5’, which is generally intelligible but dubiously English. But we will be concerned exclusively with the definite English definitions of definite sets of naturals which SP could produce, where by ‘definite’ is meant ‘non fuzzy’, i.e. that SP would regard as perfect English definitions of sets of naturals at any time. We will call them ‘SP-definitions’.

It seems that for any definable and countable set S of SP-definitions, there is at least one such definition not in S obtained by the well-known Richardian diagonal procedure: define a length-alphabetical order O and let E be the enumeration of S according to O; E is definitely definable by reference to O; consider the definition F: “the set R containing every natural number that is not a member of the set defined by the definition it is associated with in E”. If the definition F of R were in E, it would be associated with some natural number k, and k would be a member of R if and only if it weren’t. As a consequence, F is not in E and, consequently, is not in S. But F is an SP-definition.

As for indefinite extensibility, P is now the concept of SP-definition, Q is the concept of enumerability and *f* is the Richardian diagonalization function. ‘SP-definition’ is indefinitely extensible with respect to ‘enumerable’.

As above, the conclusion should be that there is no enumerable set D containing all definite English definitions of definite sets of natural numbers which SP could produce.

But, on the other hand, it seems there must be an enumeration of D because D is a subset of the countable set SL of all finite strings of letters of the English alphabet. Or at least D can be bijected with a subset of SL, namely, the set SD of the *signifiers* of the members of D, if we are to take into account the distinction between ‘signifier’ (syntactic object) and ‘signified’ (semantic object) as proposed by Saussure (1983: 139).

This contradiction is just Richard’s paradox.

I will consider two ways out and I will opt for the second.

The first proposal is the usual and argues that the concept of SP-definition is vague or fuzzy or simply non-existent, since ‘definition’ only has a definite sense when referring to definitions in some formal language.

The cost of pronouncing the concept of SP-definition non-existent seems unbearable: we are naturally compelled to accept the existence of some SP-definitions clearly recognizable as such.

I claim further that the concept of ‘definite definition in English of a definite set of naturals’ can hardly be fuzzy, because if it were, there could be some object which would be a, say, 50% definite English definition of a definite set of naturals; but this does not seem to make much sense: a definite definition which would be only ‘50% definite’ would be no definite definition at all.

Let’s represent fuzziness by means of evaluations of sentences using truth values taken from the closed interval [0,1]. Let ‘1’ denote classical definite truth and ‘0’ classical definite falsity, and let all other intermediate values denote fuzzy ascriptions of truth. For instance, ‘0.5(Pc)’ means that c is exactly 50% P. A concept/predicate P is definite if and only if for all x either 1(Px) or 0(Px), so that no sentence of the form ‘r(Pc)’, with $0<r<1$, is true.

Let P be any concept/predicate. ‘1(P)’ is the new concept/predicate ‘definitely P’. We can state: for any P, 1(P) is definite, i.e. for all x, either 1(1(Px)) or 0(1(Px)). Consider again that if ‘r(1(Pc))’, with $0<r<1$, were true, c would not be 1P (definitely P) at all, and we would also have that ‘0(1(Pc))’ is true, and this would be a contradiction. Of course, 1(P) could have an empty extension but the examples provided above show this is not the case for the concept of SP-definition.

In conclusion, the usual solution to Richard’s paradox confronts deep difficulties; deep enough as to render it sensible to consider an alternative way out.

The alternative proposal asserts that neither the set D of all SP-definitions nor the set SD of the corresponding signifiers are subsets of the set SL of all finite strings of letters of the English alphabet, so that there might be no necessity for there to exist to an enumerable set of all SP-definitions. I will try to show that D and SD are not even sets. Since the concern is primarily with SD, I will deal with this.

Richardian diagonalization provides the function we need to show that SD, if existing, is indefinitely extensible with respect to enumerability. On the other hand, it is absolutely clear that if SD exists, it has to be enumerable, since it would then be a subset of the enumerable set of all finite strings of letters of the English alphabet. Thus, we know that SD does not exist and this is ultimately why we have no reason to assume the existence of a set of all SP-definitions. What we have obtained is a Cantorian ‘inconsistent multiplicity’ that is *circumscribed* inside an enumerable set, a little in the way the infinite Koch’s curve is circumscribed inside a finite circumference. This means that being *set-like* could not be a question of size.

But the situation can be given further clarification. I wish to show how extensibility impedes the existence of SD. We argue from the assumption that the concept of *signifier of an SP-definition* is indefinitely extensible with respect to the concept of *enumerable set.* This means that the objects falling under the former can be arranged into a *hierarchy of logical levels* in the following way. We start from a definable enumeration E^1 of the signifiers of the SP-definitions that do not diagonalize on any previous such enumeration; we then take the signifier S^1 of the Richardian diagonal of E^1; S^1 is on a higher logical level than all signifiers in E^1. If we add S^1 to E^1 we obtain E^2; then we diagonalize out of it in order to get S^2 at a higher logical level, and so on. At each new logical level new strings of letters become signifiers of SP-definitions. Of course, we can obtain at each logical level as many new signifiers as different definable enumerations we can produce but the point is that we cannot take those signifiers as *given at once.*

The hierarchy of levels has as a consequence that we have no property or condition available to use the Separation Schema to detach from SD all signifiers of SP-definitions. The condition *being a signifier of a definite English definition of a definite set of natural numbers* is not enough because it doesn’t specify the level at which a member of SD is to be the signifier of some such definition. The condition *being at any* *level a signifier of a definite English definition of a definite set of natural numbers* is not valid either, because there are levels *after* it, at which new signifiers of definitions become such. For suppose there are not such levels and the condition succeeds in specifying the set; there would be a definable enumeration of that set; there would also be a diagonal definition on that enumeration, which would stand on a level higher than all the levels referred to by the condition, and this would be in contradiction with the assumption. It seems that no condition can stretch itself along all of the levels of the hierarchy and this prohibits the existence of the set we are looking for.

The fact that we are unable to specify a condition referring to all levels seems to imply that we are incapable of referring to the totality of our potential linguistic competence, not even by means of names like “English”, “French”, “human linguistic competence”, and the like. This entails that no language *L* of SP could include a syntactic device capable of avoiding this level dependence; no L could ever incorporate the level of its expressions into its signifiers so as to free the resulting expressions from this logical context dependence. For if some SP could develop such a language, Richard’s paradox would irreducibly arise in his language.

My conclusion is that taking the concept of definite English definition of a definite set of naturals as indefinitely extensible with respect to enumerability, could offer an escape to Richard’s paradox that is worth considering.

II. BERRY

There might be types of concept extensibility others than the one depicted in Shapiro et al. (2006). Human linguistic competence seems to be a somehow extensible ability in more than one respect. For example, Berry’s paradox could be pointing at some other kind of concept extensibility.

Let me introduce here the concept of extensibility *simpliciter;* it is the same as the concept of indefinite extensibility above but without the requirement in the condition 2. Now it happens that for any set S of P’s that is Q there is one f(S) that is P and is not in S but we have no guarantee that $S \cup \{f(S)\}$ is again Q. The consequence of a concept P being extensible with respect to Q is all the same that there is no set of all P’s that is Q.

Consider this version of Berry’s paradox. Let’s take any concept for whose extension there is a definable well-ordering. For instance, the concept of natural number and the relation <. This is the *Berry definition* for that concept and that ordering:

B) ‘the smallest natural number for which there is no definite English definition in less than one hundred words’

B defines no object. But exactly why? The concept of ‘natural number definitely definable in English in less than one hundred words’ is extensible with respect to the concept of ‘set definitely definable in English in less than *n* words’, where tentatively *n=93,* i.e. the 93 remaining till 100 after counting the 7 words ‘the smallest natural number not belonging to’ that must go in front of the definition of the set to get the Berry definition. Here *f* would be the function that produces the corresponding Berry definition. For any set S of natural numbers definitely definable in English in less than one hundred words, that is definable in less than *n* words, we can definitely define in English in less than one hundred words a natural number not in S.

As a consequence, there is no set of all natural numbers definitely definable in English in less than one hundred words that is definitely definable in English in less than *n* words; in particular the concept expressed by ‘natural number definitely definable in English in less than one hundred words’ has no set as its extension because obviously, if it had some, that set could be definitely defined in English in less than *n* words. Assuming the usual model theoretic doctrine that the universe of discourse for any interpreted language has to be a set, we can say that we can never speak at once of the whole extension of that concept. It is easy to see that this solves the paradox, since we cannot take B to refer to any totality of numbers in order to define the first greater than all of them.

In a similar way, it can be argued that the concept of ‘definite English definition of a natural number in less than one hundred words’ is extensible with respect to the concept of ‘set definitely definable in English in less than *n* words’, where tentatively *n*= 90, i.e. 100 minus the 10 words ‘the smallest natural number not defined by a member of’ that must go in front of the definition of the set in order to get the Berry definition. It follows that there is no set of all definite English definitions of natural numbers in less than one hundred words that could be definitely defined in English in less than *n* words; in particular the concept expressed by the 13 words ‘definite English definition of a natural number in less than one hundred words’ has no set as its extension. This solves the paradox on its own, since we cannot take B to refer to all definite English definitions of natural numbers in less than one hundred words.

Is the concept of ‘natural number definite definable in English in less than one hundred words’ fuzzy? No. As above, the adjective ‘definite’ prohibits it from fuzziness. And the concept’s extension is surely not empty. The concept is just extensible and the local consequence is here again that it has no set at all as its extension.

This entails again that ‘being too large to be a set’ is not a question of size. We wish to stress that this is no dramatic scenario. Model theory and the theory of extensibility strongly suggest that being a set-like multiplicity is just being the empty set or being a possible universe of discourse. The appearance that being set-like is being set-sized is an illusion induced by the very restricted use that has usually been made of set theoretical concepts.

It seems unavoidable to comment on the possibility that other set theoretic phenomena, such as non enumerability, might not be due to size or cardinality. This would again be no dramatic scenario, since equicardinality has become in set theory just another name for bijectability and, most probably, all we really need for foundational purposes in mathematics is the equivalence of both concepts for the finite sets, and this is beyond any doubt.

Let me conjecture that, at least in some cases, the air of fuzziness surrounding linguistic objects such as definitions could turn out to be due to some kind of extensibility rather than to actual fuzziness. This should grow clearer below.

III. TOKENISM AND LOGICAL LEVELS

Gaifman (1992, 2000) and Goldstein (1992, 2000, 2006) have suggested that some linguistic objects may possess different logical and semantical value in different logical contexts or at different logical levels. Consider the sentence-token:

(1) (1) expresses no true proposition

A double *reductio* shows that (1) is not true and is not false. (1) expresses no proposition and in particular expresses no true proposition; so we are entitled to assert the following sentence-token:

(2) (1) expresses no true proposition

Now (1) and (2) are different tokens of a same sentence-type and possess different logical values in spite that they are syntactically identical word for word. This can only be so in virtue of the different logical contexts they stand in. Those contexts seem to be hierarchical levels of reflection. Consider that we can only assert (2) after an assessment of (1) has been accomplished; we express by means of (2) the result of that assessment (or part of it), something we cannot do by means of (1) itself. (1) attempts to perform a kind of self-diagonalization in the sense that it is so defined that, if meaningful, it would have to be different from itself.

Now, consider the set SL of all finite strings of letters of the English alphabet. There are some members of SL that express SP-definitions only *after* certain level of iteration of the Richardian diagonalization procedure have been reached; this is true for the strings of letters that express SP-definitions diagonalizing out of some previously defined enumerations of SP-definitions. The phenomenon seems to be always the same: in order to avoid self-diagonalizers (in some extended sense) we must set up a hierarchy of logical levels. There exist syntactical objects that become meaningful only from some logical level on, never before.

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