THE INTRODUCTION OF LOGICAL TEMPORALITY ***DRAFT***

©Laureano Luna Cabañero
laureanoluna@yahoo.es

... alius est actus quo intellectus intelligit lapidem, et alius est actus quo intelligit se intelligere lapidem, et sic inde. Nec est inconveniens in intellectu esse infinitum in potentia ... **Th. Aqu. S. Th. I q.87 a.3. ad.2**

INTRODUCTION

In the first part of the paper an examination of Richard's paradox reveals affinity with the Strengthened Liar, so that it is worth trying to apply to the former the technique Laurence Goldstein and Haim Gaifman proposed for the latter. This seems to lead to the introduction of a new kind of hierarchy of logical levels, which I call *logical temporality* and which in turn suggests a new interpretation of some set theoretical concepts, of limitation theorems and of paradoxes. In the second part, we trace the cognitive roots of logical temporality by means of some key concepts of Husserl's Phenomenology. There I delve into the possible common root of the three mentioned logical-mathematical phenomena.

Since the first part of this paper deals with topics of mathematical logic while the second attempts a philosophical explanation by means of phenomenological tools, the second part is obviously much more speculative.

A first appendix containing a proof for the impossibility of a formal solution to paradoxes is attached. A second appendix deals with paradoxes in which no direct self-reference (and sometimes no self-reference at all) is to be found. A third appendix argues for the impossibility of l-temporal disambiguation of natural language.

PART ONE: RICHARD REVISITED VIA TOKENISM

I. The approach suggested by the title is motivated by two facts. First, an examination of the most often proposed solution to Richard's paradox (Richard 1905) reveals it as hardly admissible. Second, Laurence Goldstein's (Goldstein 1992, 2000, 2006) and Haim Gaifman's token semantics (Gaifman 1992, 2000) seems to suggest an alternative. This approach assumes as empirically evident that Richard's paradox makes sense because the names of natural languages ("English", "French", etc.) are sufficiently univocal or can be rendered so; this amounts to accepting that the expression "natural linguistic competence" and the name of the natural languages that instantiate that competence are able to define or name in a sufficiently unambiguous way. If natural languages do not contain an unambiguous core of grammatical and lexical legality (or can be so transformed as to contain it), the whole problem becomes nonsensical.

I briefly present a version of the paradox. The original version involved French definitions of real numbers while I shall refer here to definitions in English of sets of natural numbers.

Suppose an enumeration *EL* of the set *SL* of all finite sequences of letters of the English alphabet (with perhaps a finite set of auxiliary additional symbols) according to a well-defined ordering *O* by length and then by some alphabetical criterion. Such an enumeration is obviously possible and even recursive. It seems that EL must contain all expressions that are *undoubtedly* definitions in English of sets of natural numbers, that is, EL has to include an enumeration *EF* of those definitions. Nevertheless we can define in English, by reference to EL, a (diagonal) set *K* of natural numbers whose definition cannot be in EL. Let K be **the set of every natural number *n* not mapped by EL to a string of letters expressing an English definition of a set of natural numbers to which *n* belongs.** According to this definition, for any set S_n of natural numbers defined in EL by a string of letters mapped by EL to the natural number *n* we have:

(1) $$n \in K \leftrightarrow n \notin S_n$$

So K must be different from any set of natural numbers defined in EL and its definition must be different from any definition in EL. Nevertheless, we have defined K in English above (bold face) by means of a string of letters; let us call it *DD.* Now DD has to be in EL and at the same time must be different from any definition in EL. Note that K can be unambiguously defined by DD because EL can be unambiguously defined by reference to O.

II. There are only two ways out. Either DD is not an undoubted English definition of a set of natural numbers or DD is a definition that effectively diagonalizes on EF and is therefore not in EL.

If we decide on the latter, then we are led to another awkward conclusion: there must be no enumeration of all undoubted English definitions of sets of natural numbers, because if there is an enumeration of them, then there is a definable enumeration of them (by means of O) on which we can diagonalize provoking a contradiction; and this even though all such definitions seem to be contained in the set *SL*, which in turn is obviously countable. This would depict a second impossible situation, since it can be easily proven that any subset of any countable set is either finite or countably infinite.

So, the usual attempts to solve the paradox try to establish that DD is no English definition of a set of natural numbers. A widespread proposal argues that the concept of *English definition of a set of natural numbers* is not well-defined, so that its extension does not exist, no enumeration *EF* of it lies in EL and, consequently, DD fails to define anything. And the usually alleged reason to support this is that the definition of any well-defined concept of a kind of objects must make it possible to effectively decide for any given object whether that object is or is not of the kind defined. This condition was already demanded by Borel in his early comment on Richard's paradox (Borel 1908).

However, imposing this condition on any acceptable concept of a kind of objects amounts to requiring the effective decidability of the corresponding set and this seems untenable, for we usually grant the existence of definable sets that are not even recursively enumerable, e. g. the set of all total recursive functions.

Other considerations could be put forward in order to support the view that the concept of *English definition of a set of natural numbers* is ill-defined and to avoid the paradox thereby, but I will argue that those considerations would be irrelevant.

Even if there is no recursive enumeration of all such definitions, it seems that there exist some expressions which undoubtedly define in English (or according to any other linguistic code powerful enough) sets of natural numbers. Consider the chain of expressions *the set containing all natural numbers except the number zero; the set containing all natural numbers except the number zero and the successor of the number zero; the set containing all natural numbers except the number zero, the successor of the number zero and the successor of the successor of the number zero*, and so on.

English language contains a *legality* according to which some expressions *undoubtedly* are definitions of sets of naturals even if that legality cannot be expressed as a formal grammar. This legality is shared by all speakers but even if it is not quite the same in all of them, it is obvious that each (sufficiently capable) speaker is compelled to accept the existence of undoubted English definitions of sets of natural numbers. We could even substitute a private speaker and his private language for English, requiring from that speaker only that he should be consistent over time as regards his linguistic criteria. For what we are dealing with here is ultimately the human linguistic competence as could be deployed by some ideal speaker with no extra-linguistic limitations.

In addition, it is obvious that the diagonal definition of any definable enumeration of expressions that undoubtedly are English definitions of sets of natural numbers is also undoubtedly such a definition; consequently whenever we have a definable enumeration of some definitions of that kind, we have another definition of that kind not in the enumeration. Therefore, if we agree that there exist some English expressions for which it is undoubtedly a fact that they are definitions of sets of naturals, then we have to agree that no enumeration of them exists.

Note that invoking fuzziness to deny the existence of an enumeration of such definitions is not available: it cannot be fuzzy whether a particular object is *undoubtedly* an English definition of a set of naturals, because if it were fuzzy, it would be perfectly clear that it is not such undoubted definition. 'Undoubted' can be understood as 'definite', 'clear-cut' or 'non fuzzy'. If P is a predicate, then 'definitely P' is a new predicate that cannot be fuzzy, though it can indeed be empty in the sense that it may happen that no object is definitely P.

This is why declaring ill-defined or fuzzy the concept of *English definition of a set of natural numbers* looks unable to offer a solution to Richard's paradox. But, although that proposal seems untenable, one can easily understand why it was proposed and often believed to be conclusive. Richard's paradox forces us to accept the following disjunction: the set *F* of all undoubted English definitions of sets of natural numbers either does not exist or exists and is uncountable. Thus, since there is no obvious way of declaring F uncountable if existing, the easier way out was to pronounce F non existing; and, for this purpose, the handiest resort was to declare ill-defined or fuzzy the concept whose extension is F. Indeed, for a paradox to arise we do not need our definitions to form a *set*. A weaker assumption would suffice: if F could not be a set because it would be *too big* a set (as is the case with the collection of all sets) and nevertheless were the extension of some well-defined concept, the paradox would still be there: *how could a*

collection included in a set not be a set? how could a collection included in an enumerable set not be an enumerable set? That is, all we need to produce a paradox is the fact that whatever we use to express an undoubted English definition of a set of natural numbers will always be a string of letters.

The paradox can therefore be expressed in the following terms: there is a logically possible speaker *SP* developing a logically possible language *L* out of a finite set of symbols, such that there is no enumeration of all the definitions of sets of naturals SP could produce in L if SP had no extra-linguistic limitations. For if that enumeration existed, there would also exist an enumeration *ED* definable in L by reference to some length-alphabetical order *O*, and then SP could define a set of naturals by constructing the Richardian diagonal of ED, i.e. a definition in L of a set of naturals that could not be in ED. As we said, L can perfectly be a *private language* of SP in the sense that it is SP the one that decides what counts as a definition; we only have to assume that SP maintains a uniform criterion all along, which is indeed logically possible.

Now I will try to show there is a different escape from the paradox by showing that F can and must be banned on a quite different ground.

III. Gaifman and Goldstein propose solutions to the Strengthened Liar whose shared kernel, I think, suggests a way in which we could come to see why there could be no set of all undoubted English definitions of sets of naturals.

Let's examine the way they face the *meta-paradox* provoked by the Strengthened Liar. Consider the following sentence-tokens:

(2) (2) expresses no true proposition

Since no truth value can be consistently assigned to (2), (2) expresses no proposition and in particular no true one. Thus we can assert:

(3) (2) expresses no true proposition

And (3) has a different logical value from that of (2) because the logical value, i.e. the capability of expressing a true or a false proposition, is not (always) to be accorded to sentence-types but (sometimes) to sentence-tokens. (2) and (3) are indeed tokens of the same sentence-type, but they are relevantly different tokens because, in Gaifman's terms, (3) "jumps over" (2) and occurs at a "higher level" than (1), so that it can express what (2) fails to do.

Sentence-tokens, sometimes as members of a *logical network*, and not always the isolated sentence-types, are here the signifiers of propositions; such is the kernel of Goldstein's and Gaifman's proposals. This compels to distinguish *propositions*, as Saussurian (Saussure 1916) *signified*, from *sentences* as Saussurian *signifiers*, i.e. as strings of letters: within a same linguistic code, a same string of letters sometimes expresses a proposition and sometimes does not. Two identical tokens can possess different logical values if they stand, so to say, on "different logical levels".

Now back to Richard's paradox. Call DD_2 the token of the string of letters DD that defines K (i.e. **the set of every natural number *n* not mapped by EL to a string of**

letters expressing an English definition of a set of natural numbers to which n belongs), diagonalizes on EF and cannot be in EF at the "logical level L_1" to which DD_2 refers in order to define K. Call DD_1 the token of DD in EL at L_1. Call *R* the relation "being on a higher level than" that holds between (3) and (2) according to Gaifman's approach. I suggest that the same relation *R* holds between DD_2 and DD_1. That is, at the "logical level L_1", which DD_2 refers to, the string of letters DD_1 expresses no definition; at the "higher logical level L_2", on which DD_2 stands, that string of letters does express a definition. So, DD_1 and DD_2 are different *tokens* of a same expression-type. Note that, similarly to (3), DD_2 cannot refer to objects of the same "logical level" on which it stands.

Just as (2) fails to express what (3) does express, and this because of a "difference of logical level", so DD_1 fails to define what is defined by DD_2. A diagonalization out of an enumeration would take us to a "higher level", we would be "rising above" all the objects in the enumeration, much the same way (3) "jumps over" (2). DD_2 expresses a definition on a higher level than any signifier of a definition in EL to which it can refer.

So (2) tries to "jump over" itself or "diagonalize out of" itself and cannot help failing. DD_1 tries to diagonalize on itself and fails as well. But (3) and DD_2 perform at some "higher levels" what is impossible for (2) and (3) at the "levels" they belong to.

As we have to distinguish propositions from their signifiers (that is, form the strings of letters that express them), we also have to distinguish English definitions of sets of natural numbers from their signifiers, and indeed in the same sense: it is not the case that whenever a sentence-type is present, the proposition it expresses "at some level" is also present; and it is not the case that whenever a linguistic form, that "at some level" expresses a definition, is present, that definition is present too. Consequently, a definition could be absent from an enumeration or a set where the corresponding string of letters is present.

I suspect that this extensibility or distribution along "logical levels" is commonly confused with fuzziness. Linguistic objects usually employed in logic and mathematics -sentences, definitions, predicates...- may display a false appearance of irreducible fuzziness, vagueness or ambiguity, which could actually be rather the sign of their extensibility or distribution into "logical levels".

III. But we still have to confront the immediate odd consequence: the existence of multiplicities that cannot be sets, the existence of *inconsistent multiplicities* that are nevertheless *circumscribed* to sets in the sense that any eventual "element" of the multiplicity is an element of the circumscribing set. Of course, what this implies, and what has to be accounted for, is that being *set-like* is not always a matter of *size*.

In order to try to sketch an explanation I will resort to the notion of *indefinite extensibility of a concept P with respect to a concept Q* (where both concepts can be the same). This notion has been introduced by Shapiro and Wright (2006) who develop an idea that was already present in Russell and Dummett and that is closely related to Graham Priest's *Inclosure Schema* (2002, p. 133). My version is the following.

A concept *P* is indefinitely extensible with respect to a concept *Q* if and only if:

1. For any set *S* of P's that is Q there is a function *f* such that f(S) is a P and is not in S.
2. $S \cup \{f(S)\}$ is Q.

For instance, the concept of natural number is indefinitely extensible with respect to the concept of finite, since for any finite set *S* of natural numbers the function *the least natural number not in S* gives a natural number *n* not in S such that $S \cup \{n\}$ is also finite. An immediate consequence is that there is no finite set of all natural numbers. The concept of set of natural numbers is indefinitely extensible with respect to the concept of enumerable, as Cantor showed, so that there is no enumerable set of all sets of natural numbers. The concept of set is indefinitely extensible with respect to the concept of set, so that there is no set of all sets. In most cases the function *f* in 1. is a kind of diagonalization.

The concept of undoubted English definition of a set of natural numbers behaves as indefinitely extensible with respect to the concept of enumerable, being *f* the Richardian diagonalization. As a result, there is no enumerable set of all undoubted English definitions of sets of natural numbers. But, since the set of all such definitions, if existing, would have to be enumerable, the set does not exist.

The extensibility we are dealing with is, according to all appearances, extensibility along our hierarchy of "logical levels"; so we must try to outline the precise way in which the existence of "logical levels" impedes the existence of the set *F* of all undoubted English definitions of sets of naturals and of the set of the corresponding signifiers. The most essential reason we have to believe that that F exists is indeed that the set of the corresponding signifiers seems to be a subset of the set *SL* of all strings of letters of the English alphabet; so I will try to show why that is not the case.

Most probably we have no property or condition available to use Separation for separating all signifiers of undoubted English definitions of sets of naturals from the set *SL*. The condition *being a signifier of an undoubted English definition of a set of natural numbers* is not enough because it doesn't specify the "logical level" at which a member of SL is to be the signifier of some such definition. The condition *being at any logical level a signifier of an undoubted English definition of a set of natural numbers* is not valid either, because there are "logical levels" after it, at which new signifiers of definitions become such. For suppose there are not such levels and the condition succeeds in specifying the set; there would be a definable enumeration of that set; there would also be a diagonal definition on that enumeration, which would stand on a "logical level" higher than all the levels referred to by the condition, and this would be in contradiction with the assumption. It seems that no condition can stretch itself along all of the "logical levels" of the hierarchy and this prohibits the existence of the set we are looking for.

The fact that we are unable to specify a condition referring to all "logical levels" seems to imply that we are incapable of referring to the totality of our potential linguistic competence, not even by means of names like "English", "French", "human linguistic competence", and the like. This entails that no language *L* of SP could avoid this logical-level ambiguity; no L could ever incorporate the "logical level" of its expressions into its signifiers so as to free the resulting expressions from any logical context dependence. For if some SP could develop such a disambiguated language, Richard's paradox would irreducibly arise in his language.

It appears that the different types of diagonalization acting in the phenomenon of indefinite extensibility provoke jumps to "higher logical levels", whatever these are. And it seems to be immediately suggested, that set theoretic phenomena, such as enumerability or *sethood,* traditionally interpreted as related to cardinality, might rather be the effect of the distribution of logical or mathematical objects along a hierarchy of "logical levels".

IV. Admittedly, the nature of the relation *R* (i.e. what kind of things those "levels" exactly are) is still to investigate. We know that diagonalization appears to take us regularly to a "higher level", but not much more. However, the analysis of (2) and (3) as well as of DD_1 and DD_2 suggests the existence of some kind of "logical temporality", not as a strict sense chronological ordering but as a compulsory order of introduction or definition of some logical and mathematical objects: we cannot assert (3) *until* we have assessed (2) and we cannot construct DD_2 *until* we have a definition of the enumeration *EL,* in which DD_1 is contained.

If in the mathematical universe we can find sometimes an object that can be defined *only after* some other has been defined and only by leaning on the previous presence of the latter, then we can establish an ordering relation of "logical posteriority" inside some classes of mathematical objects. In some fields this "logical temporality" seems *inexhaustible,* i.e. whenever we have defined an object, we are entitled to define some other "logically posterior" to the first. This inexhaustibility could be the ultimate root of Gödelian incompleteness and of transfinite set theory. And paradoxes would be the punishment for the Babelian sin of trying to exhaust the inexhaustible "logical-temporal" ordering, that is to say, the sin an object commits when it attempts to contain its own diagonalization, so embracing the totality of the "logical levels": if we try to climb up to the top of the sky we will get our language confused. The fact that inexhaustibility arises from the need to avoid circularity could explain why limitation theorems and Cantor's theorem usually rest on *reductio ad absurdum* arguments that do not primarily come down to a contradiction like:

(4) $p \,\&\, \sim p$

but to a circularity-plus-paradox result of the form:

(5) $p \leftrightarrow \sim p$

This is the general way in which diagonalization proceeds. As Gödel wrote (Gödel 1931, footnote 14):

Any epistemological antimony could be used for a similar proof of the existence of undecidable propositions.

This suggests that the limit established by limitation theorems is needed to avoid paradox and that paradoxes just violate the limit that limitation theorems impose.

The proposal of a "logical temporality" gives rise to important questions in philosophy of mathematics. Outstandingly, the dependence of the properties of some mathematical objects on the way we can define and introduce them seems incompatible with

Platonism and hints at some kind of *definitionism.* Besides, the study of "logical temporality" would most probably lead to research on the cognitive basis of the fact that we are forced to keep such an order in the introduction of some mathematical objects; research into the *phenomenology of thinking* is likely to be required.

V. I want to briefly consider the possibility of solving Richard's paradox by means of a suitable hierarchy of formalized languages on an alphabet ALPH starting from a formalizable part ENG of ordinary English and rising through hierarchical levels up to the first non constructive ordinal ω_1^{CK}. This has been privately suggested to me by William Taylor, from Christchurch, New Zealand. Certainly, the sometimes called "Church-Kleene theorem" (Church 1938) guarantees the non existence of a formal definition of the enumeration of all formal definitions of sets of naturals in the languages of the hierarchy, and this blocks the possibility of Richardian diagonalization.

This is indeed a solution for some version of Richard's paradox referring to formal languages. But such a hierarchy could not retain the entire expressive power of natural language; if it could, the set of all undoubted English definitions of sets of naturals would exist, but it doesn't.

PART TWO: THE PHENOMENOLOGICAL ROOT OF LOGICAL TEMPORALITY

I. The Strengthened Liar paradox cannot be satisfactorily solved by Goldstein's and Gaifman's tokenism unless there is a way to make intuitive why (2), unlike (3), expresses no proposition even if it describes the same *state of affairs* which (3) describes. If we admit that (3) expresses a proposition, then we accept that there is a well defined state of affairs depicted in English by '(2) expresses no true proposition'. Consequently, we must try to explain why that state of affairs is not *available* to be asserted as being the case by (2) even if, according to all descriptive conventions, (2) is able to describe it. The following paragraphs intend to open the way to a proposal and to draw some relevant conclusions from it.

In a strict sense, a paradoxical object is any object from whose admission as a legitimate logical object (concept, set, definition, proposition, etc.) a sentence of the following form is *immediately* entailed:

(5) $p \leftrightarrow \sim p$

Immediately means that "p & ~p" need not be used to derive (5). So, simple contradictions of the form "p & ~p" are not paradoxes.

(5) represents a genuine circularity for it implies that the truth value of *p* is determined by the truth value of *p* and that there is no other way of determining it.

The class of paradoxical objects so defined contains the classical semantic, epistemological and set theoretical paradoxes.

Of course, the scheme

(6) $p \leftrightarrow p$

is sometimes associated to circularity, as in the Truth-teller:

(7) (7) is true

where the scheme (6) applies and corresponds to the fact that the *truth value of p can only be determined out of itself*. We can express this as follows:

(8) p ♥ p

(5) and (8) are the *circularity schemes.* The fact that (7), though not paradoxical itself, is obviously related to the Liar in a fundamental way reveals that circularity, not contradiction, is the essence of paradoxes.

Significantly, we find that *diagonalization*, as a typical procedure of demonstration by *reductio,* draws its proving force from the need to avoid paradox, to avoid the scheme (5) and sometimes the scheme (8). In its most general trait the diagonalization procedure is a way of proving that a certain object does not belong to a certain set because otherwise a circularity complying with scheme (5) or sometimes with scheme (8) would occur.

Diagonalization is used both in the proofs of limitation theorems and in the proof of Cantor's theorem, which stands in the core of transfinite set theory. Cantor's theorem can also be construed as a limitation theorem or a theorem of inexistence referring to the inexistence of some bijective mappings. So, as I said, the paradox seems the result of trying to break the same limits that limitation theorems reveal.

Banning circularity only makes sense when *real* processes of determination are involved (where circularity is simply impossible, so that its presence implies the impossibility of an effective determination), not purely *ideal* objects and relations. This becomes clear when we compare the concept of *logical implication* with the concept of *logical proof:* a proposition always implies itself but is never a proof of itself, and this is so because, while an implication is a purely ideal relation, a proof is a real process of determination of mental contents (even if it is based on ideal relations of logical implication), so that it does not abide circularity. In the realm of real processes a universal law establishes that nothing can determine itself; I will call this the *principle of non self-determination* (*PND*).

Hierarchies are the usual way to get around circularity in logic and mathematics. There are plenty of proposals to that effect; logical temporality (*l-temporality* hereafter) is one of them. Since l-temporality is required to prevent circularity its root should be found in some real processes of determination of our mental contents. For the examination of that kind of processes I resort to *Phenomenology*, and concretely to *psychological phenomenology* as it was proposed by Husserl (Husserl 1900, 1901, 1913, 1925a, 1925b).

According to psychological phenomenology the main feature of a psychical act is *intentionality.* An act is said to be intentional if it is directed to an object, in such a way that the act and the object are *distinguishable momenta* of mental events even if they are not *separate parts.* When we think, we always think *something,* in our mental acts there

is always something that is believed, rejected, desired, hated, remembered, thought etc. Intentional acts contain an *intentional reference* to an intentional object. In general, Phenomenology is not concerned with concrete and contingent mental states-of-affairs but only with their *essential* and *necessary* features, that are called *eidetic.* It is an eidetic feature of intentional acts that *no intentional act can be its own intentional object nor can be contained in its own intentional object*. I will call it the *principle of no intentional self-reference* (*PNS*). "Self-reference" must be understood here exactly in the defined sense, for there exist some kinds of linguistic self-reference compatible with PNS, particularly the kind of self-reference we find in the Gödel sentence in its meta-mathematical interpretation, which is ultimately reducible to a reference of a proposition to a sentence, i.e. to a mere strings of symbols, as in:

(PSEUD) this sentence contains five words

where only a pseudo-self-reference is to be found, since (PSEUD) is not about itself but about its *signifier.* Similarly, Gödel sentence, when meta-theoretically interpreted, is about a string of symbols which the interpretation takes as its signifier. Objects like (PSEUD) render dubious the attempts of solution to paradoxes by means of Tarskian linguistic hierarchies.

PNS is doubtlessly akin to the Russell's Vicious Circle Principle (VCP, Russell 1908). But, as in the case of Tarskian hierarchies, I very much doubt any such principle can hit exactly the point if it is not based on phenomenological analysis. So, for instance, there are lots of impredicative definitions that seem to break VCP and are, nevertheless, by no means in contradiction with PNS.

PNS derives from PND in the following way: since an intentional act is at least partly determined by its intentional object, if the act were its own intentional object or were contained in it, it would be determining itself. The object must be available *previously* to the act that tends toward it. For instance, no one can really think *what I'm thinking just now is right* or *what I'm thinking just now is wrong.* Such acts are *phenomenologically impossible.* Any sentence expressing a proposition has to be a linguistic objectivation of the content of an act of thinking (of a *thetic* act of thinking, as Husserl would say). If (7) and (2) expressed any propositions, they would have to be objectivations of the content of acts of the same kind I have just declared impossible; therefore they are not objectivations of the content of any possible acts of thinking and this is why they do not express any propositions. No paradoxical or simply circular sentence is the objectivation of the content of an act of thinking permitted by PNS. This suggests taking PNS not just as a descriptive thesis but also as a normative principle in order to ban paradoxes and other circularity-stricken objects. This step from the descriptive to the normative is justified by the fact that linguistic expressions receive their logical value from the propositional content of the acts of thinking whose contents they objectivate and this entitles us to reject as illegitimate those objects that correspond to no possible act of thinking.

It is a rather subtle question whether a proposition is the objectivation of an *intentional object* of a *thetic intentional act* or perhaps the proposition is the *expression* of the act itself, so that the proposition would also express *noetic* and not only *noematic* features of the act, namely, its thetic character. Since I shall not discuss this problem here, I will try to employ neutral terminology.

Now we can easily understand that two identical expressions may possess different logical values if they do not correspond to equivalent acts of thinking, i.e. to acts of thinking with the same propositional content. The intentional act corresponding to (3) has (2) in its intentional object, so that if (2) were a real proposition it would be the objectivation of the content of an act having the not-yet-existing (2) in its intentional object, which is evidently impossible. We can thus affirm that (3) is logically posterior (*l-posterior*) to (2) and that if (2) expressed a proposition it would be l-posterior to itself. The theory of logical temporality suggests that the ban on all objects that would be l-posterior to themselves, on all objects that correspond to no phenomenologically possible act of thinking, would ban *exactly* the paradoxical and the circular objects; so, this is the solution to paradoxes proposed by the theory of l-temporality.

We must add the following caution: there exist paradoxes which do not involve direct self-reference. We have a version of the Liar when a sentence A says (or seems to) that sentence B is not true and sentence B says (or seems to) that A is true. Here we find an indirect self-reference that triggers PNS: what A refers to could not be a well-defined state of affairs for A if it includes a reference to A's, because in such case it could not be an intentional object for A's thinking.

We have also the so called *infinite Liars,* Yablo's queue (Yablo 1993) and Sorensen's queue (Sorensen 1998). We face here cases in which a seemingly well-defined state of affairs is not such; these cases might perhaps exhibit ways in which we can find *non well-foundedness* without violation of PNS. Nevertheless, a way out of infinite Liars could be the simple evidence that infinite queues of thinkers are impossible. For instance in Sorensen's queue each thinker thinks that someone behind himself is thinking an untruth. That thought seems to refer to a perfectly well-defined state of affairs; however this cannot be so if the queue is infinite; thus we might conclude that the queue cannot be infinite.

We will meet these two kinds of paradoxes in Appendix II.

We shall come below to a definition as rigorous as possible of the relation of logical posteriority (*l-posteriority*), but for all we can say at the moment it seems that PNS is the ultimate root of l-temporality, in so far l-temporality has to be introduced only to avoid the violation of PND and that PNS is an application of PND to intentional acts of thinking. This suggests that whenever a circularity (and thence a violation of PND), or the need to avoid it, is involved in a certain logical-mathematical phenomenon, then l-temporality is the ultimate root of the phenomenon. This will be my main thesis and I wish to explicate it as follows:

(MT) every logical-mathematical phenomenon in which a circularity, or the need to avoid it, is implied has its root in l-temporality

where *circularity* must be understood as defined by the circularity schemes.

MT is the basis for the belief that paradoxes, limitation theorems and transfinite arithmetic have a common origin in l-temporality. Consider that l-temporality seems *inexhaustible,* that in some realms it happens that whenever we have defined an object we can always define a new one logically posterior to the first. As I said in part one, l-

temporality and its inexhaustibility could be the root of transfinite set theory and limitations theorems, while the sin of trying to violate l-temporality or to exhaust the inexhaustible would merit the Babelian punishment of paradox.

II. The impossibility of an object being l-posterior to itself serves as a key for the prevention of paradoxes and for the interpretation of limitation theorems. In its first use this impossibility is closely related to Rescher's *Successful Introduction Principle* (Rescher 2001 p.164). Rescher (Rescher 2001, p. 165) writes:

If an item is to be introduced into the agenda of discussion meaningfully, then its introducing specification must not presume that this item is ***already*** *available for consideration.*

In any case we must try first to define the relation of l-posteriority as rigorously as we can. We will find that it seems impossible to define it in purely phenomenological terms (in terms of PNS and intentional acts of reference), even if it is clear that l-temporality has a phenomenological root.

Provisionally, we can say that an object Z is l-posterior to another object X if and only if Z can **only** be introduced by means of an intentional act whose intentional object *explicitly* or *implicitly* contains X.

An object is said to be *explicitly* contained in an intentional object if it is *individually* present in it, and is said to be *implicitly* contained when it belongs to a certain multiplicity that is in turn explicitly contained in the intentional object. For instance, (2) stands explicitly in the intentional object of the act corresponding to (3) while DD_1 is only implicitly present, as an element of a set, in the intentional object of the act corresponding to DD_2**.** We usually introduce logical-mathematical objects by means of definitions and these are linguistic objectivations of the contents of intentional acts.

One could wonder why PNS applies to objects that are only implicitly present in an intentional object. To get this explained, one has to consider that an intentional reference to a multiplicity, even if accomplished by means of the intensional content of a concept, has an extensional intention and this implies that any element of the multiplicity at issue must, in principle, be capable of being individually given to the intentional act, even if it is not actually given so.

However, cases can be found that prove the definition above insufficient. Three circumstances force us to modify it.

First, the relation of l-posteriority has to be *well founded,* i.e. there must be objects that are l-posterior to none. This entails the existence of some logical-mathematical objects and facts that are, so to say, *l-intemporally* given. For instance, in arithmetic the field of the l-intemporal might well coincide with the field of *recursive arithmetic*. Most important is that every l-intemporally given object is independent of our way of introducing or defining it, so that the definition above is not applicable to it. We can make this patent if we consider Gödel's sentence G and compare it to Henkin's sentence H which, in its meta-theoretical interpretation, asserts that H is provable in the arithmetical system P. We can use the fact that G is interpretable as speaking about itself and about the deductive power of P to show that G is l-posterior to P and hence

not provable in P if P is sound. In the same way the Liar tries to diagonalize out of itself and fails, G succeeds in diagonalizing out of the deductive power of P. So G behaves in relation to P just as the Liar tries to do with regard to itself. On the grounds of an evident parallelism one could expect that H would also behave in relation to P as (7) tries to do with regard to itself. But this is not the case: Löb's theorem (Löb 1955) shows that H is provable in P and so that it does not diagonalize out of it at all. The cause of this difference is the fact that the Gödelian predicate BEW is only *weakly representable* in P (or in any sound system able to compute recursive arithmetic), which implies that if a formula is provable in P, then P proves it is so, while it is not the case that whenever a formula is not proved in P, then P proves it is not. H is a Σ_1 formula while G is a Π_1 formula. We could say that only the affirmative use of BEW belongs to recursive arithmetic (though this is not the usual meaning of *recursive arithmetic*); BEW is a Σ_1 predicate and therefore recursively enumerable even if it is not recursive. But this entitles us to conjecture that the provability of H in P is an l-intemporally given fact.

Second, it seems natural that logically equivalent objects possess the same l-temporal properties: the relation of logical equivalence would *export* the l-temporal relations through non phenomenological channels from some objects to some others. Thus any object logically equivalent to some object which is in turn l-posterior to some other, must be also l-posterior to the third object.

Third, the relation of l-posteriority, as an ordering relation, has to be transitive. Thus any object l-posterior to some object which is in turn l-posterior to some other, must be l-posterior to the third object too.

The distinction between the real and the ideal is a central topic in Phenomenology, and it appears that in the shaping of l-temporality the ideal functions of thought (such as logical implication or subsumption under concept) are intricately mixed up with the real function of intentional reference, and this is what makes impossible a definition in pure phenomenological terms.

Taking these three circumstances into account, we can give the following general but still provisional definition:

(GD) An object Z is l-posterior to an object X if and only if at least one of the following conditions is fulfilled:

1st. Z can only be introduced by means of an intentional act whose intentional object explicitly or implicitly contains X, and Z is not l-intemporally given.

2nd. Z is logically equivalent to some object which is l-posterior to X.

3rd. Z is l-posterior to some object which is l-posterior to X.

When the l-posteriority relation between Z and X does not depend on the transitivity clause, we say that Z is *immediately* l-posterior to X, and we write it so:

Z / X

In general, that Z is l-posterior to X is written so:

$$Z \,//\, X$$

I must insist on the fact that the definition above is a provisional proposal. In any case note that l-posteriority need not be a linear ordering.

III. The intentional references in which the object referred to is only implicitly present (as a member of an explicitly present multiplicity) in the intentional object are achieved by means of the referring power of concepts. Any multiplicity whatsoever that is able to be explicitly present as such in an intentional object has to be *definable*, and it is always by means of a concept (i.e. *intensionally*) that a multiplicity gets defined as such; otherwise only the members of the multiplicity but not the multiplicity itself would be explicitly present. When we refer to some objects by means of a concept we go through its *intension* down to its *extension* or part of it.

Consider that the width of the referring power of a concept is exactly the *extension* of the concept. In addition, paradoxes show us that the members of the extensions of some concepts cannot be all on the same l-temporal level, in the same *l-instant;* so, the members of the extension of the concept of *token of the Strengthened Liar*, the members of the extension of the concept of *English definition of a set of naturals* or simply the members of the extension of the concept of *set* have to be introduced along a hierarchy of stages that, I claim, are nothing else than *l-instants.* I call such extensions *lt-distributed extensions.* As a consequence, it is clear that the referring power of some concepts is correspondingly lt-distributed.

If C is a concept, let $\hat{C}$ be its extension; let's express that $\hat{C}$ is lt-distributed by means of the following symbols:

$$/\,\hat{C}\,/$$

We usually interpret that when Epimenides the Cretan wrote something like *all statements by Cretans are false* he made a false move for he tried to make both a referential and an l-intemporal use of the concept of *statement by a Cretan*, whose extension is lt-distributed. It must be noted that whether Epimenides's words actually committed an attempt at self-diagonalization or it is rather the fault of the interpreters who read 'all statements by Cretans' as necessarily referring to Epimenides's statement itself, depends on linguistic conventions on the use of the quantifier 'all'; concretely, the issue is whether 'all' must be interpreted as l-intemporal or it should always be read as restricted to what is l-prior to the l-instant in which the quantifier is used.

The latter is perhaps an optional way (strongly suggested by l-temporality theory as we will see) towards the following general conclusion:

In a given l-instant we can only refer through the intension of a concept to objects l-intemporally given or given in some logically prior (*l-prior*) l-instant.

Indeed, this is only the application of PNS to the intentional references that are executed through the intensional content of a concept. If, in the act of constitution (i.e. definition or introduction) of an object, the very object we are trying to constitute were subsumed

under the concept we are using for intentional reference, this object would be circularly constituted and therefore l-posterior to itself. *This means that l-temporality prohibits us from using the function of subsumption under concept to outgrow the capacity of intentional reference as constrained by PNS.* When we try to overcome the limits of intentional reference by leaning on the power of concepts we generate paradoxical or simply circular objects. L-temporality serves, so to say, as an arbitrator between the clashing powers of the real and the ideal functions of thought.

Now it is a natural move to propose that whatever cannot be an intentional object for a thinker cannot be an *available state of affairs* for him to assert. The state of affairs that (2) expresses no true proposition is not available as an intentional object for the one who is uttering (2); consequently, it is not available for him as an assertible state of affairs. This establishes a principle of *l-temporal relativity in the availability of the states of affairs* (PST). This principle clarifies why (3) but not (2) is able to express a proposition.

The restriction on *concept mediated reference* implies that the extension of a concept is lt-distributed whenever we cannot refer through it to the totality of its extension because, if we could, we could also make use of that referring power to constitute an object that would belong to that very extension, so that we would also be referring to the outcome of our very act of reference and so performing an intentional self-reference. At this point the referring power of a concept has to be lt-distributed; so, the referring power of some concepts varies along l-temporality. But this is as much as saying that the extension of those concepts varies along l-temporality and has to be lt-distributed as well: *what is in the extension of some concepts depends on the current l-instant.* The lt-distribution of the extension of a concept is in turn the same as the lt-distribution of the applicability of that concept: some concepts become applicable to some objects only "from some l-instant on".

For example, the sentence-type of (2) and (3) does not fall under the concept of expression of a proposition in the l-instant corresponding to (2), but it does so in the l-instant corresponding to (3); the expression-type of DD_1 and DD_2 expresses no English definition of a set of naturals in the l-instant in which DD_1 stands but it "already" expresses some in the l-instant corresponding to DD_2. And when we are considering a plurality of objects under the concept of set, that plurality is "not yet" a set and this is why we always fall short when trying to enclose all sets within the limits of a set.

So, we can say that the bearers of the logical value are not always the expression-types l-intemporally considered but the expression-types as assigned to some l-instants; and this ultimately reduces to the fact that the bearers of the logical value are the thoughts, as intellectual contents of the acts of thinking, and that thoughts are lt-distributed; that is to say, sometimes they are only possible in certain l-temporal context. This is what ultimately lies behind the need of taking into account the *tokens* and not only the *types.*

The introduction of the lt-distribution is expressed by the following formula:

(PLT) $$\forall C\,(\sim/\hat{C}/ \rightarrow \hat{C}//\hat{C}) \rightarrow /\hat{C}/$$

which simply asserts that for any concept C if the fact that the extension $\hat{C}$ of C is not lt-distributed implies that $\hat{C}$ is l-posterior to itself, then $\hat{C}$ is lt-distributed. This is the *principle of lt-distribution* (*PLT*).

So, in the l-instant Epimenides uttered *all statements of Cretans are false,* his utterance was not subsumed by the concept of *statement of a Cretan*. As I said above, whether his utterance is paradoxical or isn't depends on linguistic conventions: if we agree that the word *all* is to be used in an l-intemporal way, then Epimenides's words tried to break the l-temporal order and, as a consequence, uttered a meaningless expression. Nevertheless, the theory of l-temporality appears to imply we cannot always demand an l-intemporal use of the quantifiers. The possibility that it is not that we *should not* but that we *cannot* use them always in an unrestricted way is strongly suggested by the theory of l-temporality. Let's say that PLT is obviously related to Dummett's idea of *le concepts* (Dummett 1963); how close that relationship is cannot be discussed here.

This can also be exemplified using what we shall call the *unrestricted diagonalizer theorem* (Thomson's theorem, Thomson 1962):

(9) $\forall R \sim\exists x \forall y\, Rxy \leftrightarrow \sim Ryy$

that is, *for any relation R, there is no object x such that for any object y, x is in the relation R with y if and only if y is not in that relation with itself.* This means there is no unrestricted diagonalizer in the sense of a diagonalizer belonging to the same class of objects it diagonalizes out of.

Assume there is such an *x* for some relation R; then by quantifier elimination we would have:

(10) $Raa \leftrightarrow \sim Raa$

Obviously *x* should diagonalize out of the set of the *y*'s. If you take R the relation of *being about* and let the variables in (9) range over proposition, (9) implies that there is no proposition about all the propositions that are not about themselves; any purported such proposition would diagonalize out of the class of propositions it is about so that that class would not contain all propositions not about themselves. Consider the following example:

(11) all propositions not about themselves are about something else

In symbols:

(12) $\forall x \sim Rxx \rightarrow \exists y\, y{\neq}x \,\&\, Rxy$

We cannot understand the universal quantifier in (12) as ranging over a domain containing (12) itself for, were it so, we would have:

(13) $R((12), (12)) \leftrightarrow \sim R((12), (12))$

Accordingly the domain of propositions which (11) refers to cannot include (11). So (11) is not about itself and thus, contrary to all appearances, (11) is not about all

propositions not about themselves but only about those propositions not about themselves to which (11) is l-posterior. The extension of the concept of proposition does not contain (11) in the l-instant in which (11) stands; we cannot use that concept in (11) to refer to what is *not yet* given.

So, sometimes, when we say 'all P are Q', we cannot refer to all P but only to those that are *already given,* in the l-temporal sense, in the l-instant which we speak from. There is a restriction in the extension of the concept of P, if its extension is lt-distributed, which in formal languages manifests itself as a quantifier range restriction.

Yet, it seems we can accord (11) an l-intemporal meaning if we interpret it intensionally, i.e. disregarding any referential use of the concept: the concept of proposition not about itself implies the *nota* of proposition about something else. But if we extend the concept of *aboutness* in such a way that a proposition about the concept of proposition not about itself is about all propositions not about themselves, then resorting to intensional interpretations does not help and it seems we are left with no possibility of speaking about all propositions not about themselves in the extended sense.

Consider:

(14) the concept of proposition not about itself in the extended sense implies the *nota* of proposition about something else

¿Is (14) about (14) in the extended sense? If so, (14) would be an unrestricted diagonalizer.

It happens that some concepts involve in their intensional contents a mention to their extension or applicability itself and this forces us to lt-distribute those intensions according to the distribution of the extensions. Consider the *concept K of concept that does not subsume itself.* The concept of house is not a house and consequently does not subsume itself. The concept of concept is a concept and consequently seems to subsume itself. But what about K? We are once more doomed to circularity because K is again an unrestricted diagonalizer.

Those concepts are inadmissible and must be replaced with lt-distributed families like: the concept K_0 of concept that does not subsume itself at l-instant 0, the concept K_1 of concept that does not subsume itself at l-instant 1, etc. where any K_n stays in l-instant n+1. That is, we substitute a family of restricted diagonalizers for the purported universal diagonalizer.

When we extended the concept of aboutness we granted the possibility of a proposition being about X only by mentioning a concept which subsumes X; so we included the applicability of a concept in the concept of aboutness. Now the concept of proposition not about itself in the extended sense implies the (non) applicability of that very concept.

This takes us still further. If we could speak at a time about all l-instants of l-temporality, then we could reproduce the paradox of K by introducing the concept of concept that does not subsume itself in any l-instant. So, we cannot speak about all l-

instants. Whenever we speak we stay in a particular l-instant and we can only refer to the l-previous l-instants from the new l-instant we inaugurate by referring to them.

But when we say

(15) we cannot speak at once about all l-instants

we seem to have achieved exactly what we have pronounced impossible; we seem trapped in a kind of Wittgensteinian (Wittgenstein 1921, §7) paradox: *wovon man nicht sprechen kann, darüber muß man schweigen.*

In fact, the resulting general rule is that whenever we speak, we inaugurate a new level of *aboutness* which we cannot be speaking about. That is, the *aboutness* performed and the aboutness referred to in (15) are not the same; let's call them respectively 'about$_1$' and 'about$_0$'. Thus, in (15) we do not really refer to all the levels which we cannot speak about at a time: we were speaking about$_1$ the levels which we cannot speak about$_0$. There are in addition some levels which we cannot speak about$_1$. And so on.

And this is why the theory of l-temporality can never be entirely exposed: we cannot refer to the l-instants not yet given in the l-instant from which we speak in each case and there is no l-intemporal stage from which we could refer to them. We can always formulate the theory from some new and l-posterior l-instant. So, the theory is *mystical* in Wittgenstein's sense (Wittgenstein 1921, §6.522):

There is indeed the inexpressible. This ***shows*** *itself, it is the mystical.* (According to C.K. Ogden's translation)

Consider this version of (15):

(16) there exist some l-instants l-posterior to the one in which I stand just now

If (16) does not refer to the l-instants that are l-posterior to the one in which (16) stands, which does it refer to? Most probably it refers to nothing at all: the problem lies firstly in the adverb 'just now', which attempts a phenomenologically impossible reference. (16) is not a part of the theory of l-temporality, though (17) is indeed:

(17) for any l-instant there are l-posterior l-instants

But there is no l-instant from which we could quantify over them all. Moreover, if we try the intensional version of (1):

(18) the concept of l-instant implies the *nota* of being l-previous to some l-instants

we must confront the fact that not all l-instants fall under the concept of l-instant in the l-instant in which (18) is on each occasion asserted.

No less *mystical,* again in Wittgenstein's sense, is the principle of Bivalence:

(BIV) all propositions are either true or false

It doesn't help if we try to render (BIV) the expression of a purely intensional relation as in:

(BIV') the concept of proposition implies being either true or false

We are obliged to avow that no proposition *p* is in the extension of the concept of proposition as used in *p*; otherwise we could use the concept to perform an intentional self-reference, thus violating PLT. The analysis of the *unrestricted diagonalizer theorem* seems to reveal that no proposition can be in its own universe of discourse.

In order to draw some general conclusion concerning paradoxes, let us first define *diagonalizing;* given a set S, *diagonalizing out of* S consists of defining an object K_S such that:

1. There is a class S^+ such that $S \subset S^+$ and $K_S \in S^+$.
2. K_S can be used to show that $S \neq S^+$ because it is in such a way defined on the elements of S that, if K_S belonged to S, K_S would be ill-defined by virtue of circularity.

The main logical paradoxes are attempts at self-diagonalization and must be avoided by suitably restricting the universe of discourse in each occasion, so that it does not include the diagonalizer. This implies a restriction in each occasion of the domain on which an intentional reference can be performed. For natural languages this implies a restriction on the referential use of concepts; for formal languages it implies a restriction on the domain of quantifiers (see about this Rayo and Uzquiano 2006).

PLT implies, as we have seen, a restriction on the possibilities of reference through a concept in a given l-instant and, since the extension of a concept is quite the same as its capability of being used for referential purposes, PLT also implies a restriction on the comprehension axiom. We can enunciate this restriction as an *lt-comprehension principle* (LTC) by means of the following formula:

(LTC) $\forall C x \alpha \beta \; \exists \hat{C} \; ((x_\alpha \in_\beta \hat{C} \leftrightarrow C(x_\alpha)_\beta) \; \& \; (C(x_\alpha)_\beta \rightarrow \Psi(\beta,\alpha)))$

which reads: for any condition C and any object *x*, constituted in l-instant α, there is a set $\hat{C}$ to which *x* belongs in l-instant β if and only if *x* satisfies C in l-instant β, and if *x* satisfies C in l-instant β, then from l-instant β it is possible to accomplish an intentional reference to any object constituted in l-instant α. Note that the relation Ψ (inspired by Dedekind's function φ, Dedekind 1888) is a specifically phenomenological object.

This formula contains an lt-distributivity of *predication,* which can be abridged as:

(LTP) $\beta(\alpha) \rightarrow \Psi(\beta,\alpha)$

where "$\beta(\alpha)$" means "β is applicable to α" in the sense of "a predicate constituted in l-instant β expresses a proposition when applied to an object constituted in l-instant α".

Of course, these statements are subject to the general l-temporal restrictions on the possibility of referring at once to all l-instants.

When the members of an extension are scattered along the totality of the recurrence of l-temporality, that extension cannot be conceived of as a single object, as a *set*. The reason is that, once we have a set S as an individual object, we can always define objects l-posterior to S (for instance P(S)), so that if the elements of S are to be found everywhere in the recurrence of l-temporality, then there will be elements of S l-posterior to S. Since to avoid circularity in the constitution of a set its elements must be l-prior to it, we would have that S is l-posterior to some objects that are l-posterior to S and thence, by transitivity, S would be l-posterior to itself. This is how the sets that would provoke set theoretical paradoxes can be banned by the theory of l-temporality.

IV. Logical temporality seems to result from the tension between the capacity of the *ideal* function of subsumption under concept and the capacity of the *real* function of intentional reference. This is ultimately a tension between concept intension and concept extension because an object is logically subsumed under a concept just in case it has the features comprised in the concept's intension but may be referred to by means of a concept only if, in addition, it is presently a member of its extension.

Even in (BIV') is not subsumed under the concept of proposition in l-instant in which (BIV') is introduced, it evidently falls under that concept in some l-posterior instant in which we can have (BIV') as an object and are no longer producing it. Then we can formulate l-posterior versions of (BIV') which apply to l-prior ones. The case is alike to that of the *concept of concept:* the formulation we are producing in a particular l-instant is not subsumed by itself, but we can always produce new l-posterior formulations that subsume the l-prior ones. The point here is that we do not interpret the l-temporally successive formulations of the concept of concept or of the concept of proposition as *different* concepts. The l-temporal division of pseudo-concepts into different concepts seems only necessary (to avoid paradoxes) when the purported concepts include as *notae* the features of reference, extension or application, which are primarily affected by lt-distribution. But even if this distinction turned out to be wrong, we could speak of the tension between the l-temporal and the naïve (l-intemporal) conception of concepts.

The tension becomes patent in the already mentioned phenomenon of *indefinite extensibility:* concept intension remains the same through a series of increasing concept extensions along l-temporality, so that subsumption is not given all at once but develops along l-temporality's recurrence; the present referential extension can never match the logical extension of subsumption; the concept's capacity of reference is at no l-instant up to its capacity of subsumption.

Anyway, the tension between subsumption and reference is to be solved in terms of use by making the former conform to the latter, and this is in turn accomplished by PLT. So, in the innermost core of the theory of l-temporality nests the impossibility of overcoming the limits of intentional reference by means of the power of concept. The theory intends to prevent a certain undue use of the purely ideal functions of reason; it tries to prevent the abuse of reason in the field of logical and mathematical thinking.

V. The theory of l-temporality must be conceived of as a *phenomenological metalogic*, since it tries to clarify some logical phenomena from a phenomenological perspective. Nevertheless, this kind of study implies by no means an attempt of reduction of ideal validities to psychological matters of fact; indeed, the ideal validity or *transcendentality* of the logical laws remains unscathed.

Certainly, the theory attempts to derive some ideal facts that can be proven as mathematical or logical truths of pure reason from a phenomenological principle, namely PNS. However, PNS is not a contingent psychological fact but an *eidetic* feature of intentional acts and hence a *necessary* truth. This explains how logical-mathematical facts, proved as theorems, can have their root in phenomenological features of our intentional acts: in any case *necessary facts are derived from necessary facts.* This is why I dare found logical and mathematical facts on phenomenological ones. As Dieter Lohmar (Lohmar 2002, p. 237) puts it:

If a concept violates the eidetic laws of constitution then the object meant cannot be given intuitively and thus the object cannot exist. In this way the criterion of reasonable motivation limits the freedom of conceptualization in mathematics.

As I said in part one, the theory gives rise to a number of philosophical questions and technical difficulties that are to be dealt with elsewhere.

APPENDIX I: IMPOSSIBILITY OF A FORMAL SOLUTION TO PARADOXES

I. A procedure is *formal* in a strict sense if it is only based on the examination of physical characteristics of some alphabet's symbols, without appealing to any other mental processes. The formal part of a natural language is the realm of what, in well-known Saussure's terminology, is called the *signifiant* (the *signifier*) in opposition to the *signifié* (the *signified*). The realm of the signifier is supposed to be ruled by some algorithms or formal axiomatic systems, which in mathematical linguistics are named "grammars". Consequently, a formal solution for paradoxes should be a *grammatical* one; that is to say, a grammar able to eliminate all paradoxical expressions of a natural language and only them; this implies rendering algorithmically decidable the set of all non-paradoxical expressions. And we have to refer to a natural language if we are speaking of a solution to all paradoxes; for formal languages lack the expressive power of the natural ones, and many of them are designed with the intended aim of avoiding the appearance of the already known paradoxes.

II. There exist some expressions that constitute paradoxes in themselves, on the basis of their *intrinsical nature*, independently of any context. I shall call those context-free paradoxical expressions *n-paradoxes.*

For n-paradoxical expressions there is no *state of affairs* able to accord them a truth value. Let us attend to the following expression:

(1') the first expression generated by ALG is false.

where *ALG* stands for a complete description of an algorithm that generates (1') as its first expression. The self-reference of (1') and, at the same time, its condition of being paradoxical, is already determined by its content: as far as (1') contains a full description of ALG, its self-reference is already mathematically settled by its very content, so that any contextual circumstance can be disregarded for the question. Consider now:

(2') (3') is true.

and suppose the expression alluded to is:

(3') (2') is false.

a circumstance that cannot be derived from the content of (2'). So, (2') turns out to be *extrinsically* self-referent in an indirect way, and thus, *extrinsically* paradoxical. In fact, it would have sufficed to name (3') another expression with a different content, to make (2') able to receive a truth value. So, (2') is by no means *n-paradoxical*. For this topic see Kripke 1975.

Trying to find a formal solution for *circumstantial* paradoxes would be nonsensical. This is why the following proof relates only to n-paradoxes. It will be shown that the set of all non n-paradoxical adjective expressions in some language SL with the expressive power of a natural language is not algorithmically enumerable. And if it is so, then there is no formal solution for paradoxes.

III. The proof is based on the possibility of *diagonalizing out of* any enumeration containing only non paradoxical expressions by means of a suitable "heterological"; it proceeds as follows.

DEFINITION 1: A sentence S is an n-paradox in the language SL if and only if there exists no state of affairs *st* such that if *st* were the case, *st* would make S true according to the interpretation of S in L.

DEFINITION 2: An expression in SL is n-paradoxical if and only if the mere admission of it as meaningful entails an n-paradox in SL.

DEFINITION 3: A *list* is an algorithmic enumeration.

PROPOSITION 1

There exists no list LE containing every non n-paradoxical expression in SL and no n-paradoxical one.

PROOF

Suppose LE exists.

I define now the adjective "heterological" in the following way: an expression is heterological if and only if it is an adjective expression in LE and yields no true sentence when predicated of itself. Note that the definition of heterological needs to describe LE and this, since LE is infinite, can only be done by means of the description of an algorithm enumerating LE.

It is easy to see that "heterological" is different from every adjective expression A in LE; from the definition of "heterological" we have:

(4') $$\forall A\ H(\text{`}A\text{´}) \leftrightarrow A \in LE\ \&\ (\sim A(\text{`}A\text{´}))$$

where "H" stands for "heterological" and "`A´" is the quotation of "A"; in (4') we allow for any adjective expression to work as a predicate.

If H were in LE, it would give rise to the n-paradox "H(`H´)" because from (4') we would have:

(5') $$H(\text{`}H\text{´}) \leftrightarrow \sim H(\text{`}H\text{´})$$

So, if H were in LE, H would be n-paradoxical and would not be in LE. So H is not in LE.

But H is not n-paradoxical because "H(`X´)" is never n-paradoxical. If X is no adjective expression in LE, then "H(`X´)" is false; if X is in LE, then X is different from H, and "H(`X´)" is not n-paradoxical because H can only raise a paradox when predicated of itself. To see this consider that, for a sentence *s* to be paradoxical, it has to happen that, if *s* said anything, then *s* would be equivalent to its negation. Consider that for all X,

“H(`X´)” is equivalent to “~X(`X´)”, and this cannot be equivalent to “~H(`X´)” if X is different from H.

Thus “H(`X´)” is never n-paradoxical; thence H is a non n-paradoxical expression absent from LE

Since there would always be a non n-paradoxical expression in SL not in LE, LE does not exist as it was defined. ■

PROPOSITION 2

There is no formal solution to paradoxes in L.

PROOF

A formal solution to paradoxes in SL would render decidable, and thence algorithmically enumerable, the set of all non n-paradoxical expressions in SL; but that would contradict proposition 1. ■

REMARK: Assume the set of all non n-paradoxical expressions in SL exists and call it *S*. If there is an enumeration of S, then there is a definable one, because we can always get a definition introducing among the elements of S a well-defined alphabetical order. As a consequence, a similar argument can be used to show that S is not enumerable. This leads to a version of Richard’s paradox; this version of the paradox shows that the set of all meaningful expressions of any language as expressive as natural language (if there is such a set) either is not well defined or is not a set, and in any case is not algorithmically enumerable.

APPENDIX II: PARADOXES WITHOUT DIRECT SELF-REFERENCE

I will examine here two paradoxes: the Card paradox and Yablo's paradox. Both are versions of the Liar containing no direct self-reference.

We can formulate the Card paradox in the following way. Suppose John thinks:

(1'') what Peter thinks is false

And suppose Peter thinks:

(2'') what John thinks is true

There is no way of consistently assigning a truth value to (1'') or (2''). But there is no direct self-reference in none of them. This seems to pose a problem for the phenomenological interpretation of paradoxes based on PNS; the state of affairs John is "trying" to assert in his thought is the following:

(3'') Peter thinks something and this (which Peter thinks) is false

Formally:

(4'') $\exists x\ Th(P, x) \& F(x)$

What (4'') expresses is a well defined state of affairs: either Peter thinks something false or he doesn't. Furthermore, it seems that having it as an intentional object does not force John to have its own thought as its own intentional object. So, the question is: why is that state of affairs not available as such for John? It could happen that the circularity evident in this situation (and seemingly responsible for that impossibility) cannot be reduced to an attempt of violation of PNS.

This is not the case. It is obvious that John and Peter could not perceive the other's thought as an intentional object because if they could, then each of them would be able of perceiving his own thought as an intentional object, thus violating PNS. Let's call *p* what Peter thinks. John cannot have as an intentional object that Peter thinks p, nor that Peter thinks p and that p is false. That Peter thinks p and p is false is formally:

(5'') $Th(P, p) \& F(p)$

It must be remembered that, according to PST whatever cannot be an intentional object for a thinker cannot be an *assertible state of affairs* for him. So, on the grounds of PNS, the expressed in (5'') is not an available state of affairs for John; this suggests that the introduction of the existential quantifier in (5''), that results in (4''), could also yield a state of affairs non available as such for John. I will painstakingly show this.

Let $ST^J[x]$ mean that *x* is an available state of affairs for John and let $ST^P[x]$ mean the corresponding for Peter.

That Peter thinks something false implies he thinks something of propositional nature:

(6'') $\exists x\ Th(P, x) \& F(x) \rightarrow \exists x\ Th(P, x) \& Pr(x)$

So it is obvious that

(7'') $ST^J[\exists x\ Th(P, x) \& F(x)] \rightarrow ST^J[\exists x\ Th(P, x) \& Pr(x)]$

But that Peter thinks something propositional implies that it is a well defined state of affairs for Peter that John thinks something true:

(8'') $\exists x\ Th(P, x) \& Pr(x) \rightarrow ST^P[\exists x\ Th(J, x) \& T(x)]$

Consequently, we have:

(9'') $ST^J[\exists x\ Th(P, x) \& F(x)] \rightarrow ST^J\ [ST^P[\exists x\ Th(J, x) \& T(x)]]$

That is, for John is has to be an available state of affairs that the truth value of John's thought is an available state of affairs for Peter. But, according to PNS, the very existence of John's thought is not an available state of affairs for John; therefore it cannot be an available state of affairs for John that the truth value of John's thought is an available state of affairs for Peter:

(10'') $\sim ST^J\ [ST^P[\exists x\ Th(J, x) \& T(x)]]$

And by Modus Tollens in (9''):

(11'') $\sim ST^J[\exists x\ Th(P, x) \& F(x)]$ ■

Of course we can reason in a similar way about Peter.

This shows that the non-availability of the states of affairs in the Card paradox can be derived from PNS.

Contrary to any appearance neither to John nor to Peter is here possible a propositional intention because John's (or Peter's) thought can only refer to what stands in an l-prior instant and is so available for it as a state of affairs. Let's now take that John, while standing in place S^J and at time T^J, says that whatever Peter says in place S^P and at time T^P is false; let's also take that Peter says in S^P and at T^P that whatever John says in S^J and at T^J is true. The only claim John can be really positing is that whatever is said by Peter in S^P and at T^P and **is available for John as l-prior** is false; indeed John is materially impeded by PNS from thinking otherwise. As in the case of Epimenides, their utterances result in truth value gaps just because either they do not express themselves in the logically correct way or we do not interpret them correctly. So the *thinker-version* of the Card paradox does not really exist.

In Yablo's paradox we consider an infinite queue of persons each of them saying:

(12'') each of my successors says something false

There is no way of according a truth value to the sentence of the first person. Here we face no self-reference either direct or indirect. So, PNS cannot account for this paradox.

The problem is now: why is the expressed in (12'') no available state of affairs for the first member of the queue?

The problem lies in the assumption of an infinite queue of so related sayers. That assumption implies the possibility of *non well-founded* states of affairs. The paradox dissolves as soon as the queue becomes finite. But it seems that if the infinite were possible, then what (12'') expresses would be a well defined state of affairs for any utterer in the queue; since this is not the case, the infinite queue seems impossible, and this is what Yablo's paradox shows.

Unlike in other cases, there does not exist here a well defined state of affairs that is not available to someone; it happens here that something that according to the law of excluded middle is a well defined state of affairs (because either everyone behind is lying or that is not the case) turns out not to be so if the queue is infinite and only then. So, either we reject the law of excluded middle or we reject the possibility of an infinite queue.

It seems an easy choice.

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